8. Qubitization: Basics

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Quantum Signal Processing

- The precursor to qubitization is Quantum Signal Processing (QSP)
- Physical model: Always-on magnetic field in one direction + instantaneous pulses

$$e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = -2\cos^{-1}(a)$ and

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1.$

Qubitization: Unitary Encoding

$$e^{i\phi_0 Z}e^{i\theta X}e^{i\phi_1 Z}\cdots e^{i\theta X}e^{i\phi_d Z} \to e^{i\phi_0'\tilde{Z}}U(H)e^{i\phi_1'\tilde{Z}}\cdots U(H)e^{i\phi_d'\tilde{Z}},$$

where $\tilde{Z} = Z_a \otimes I_s$ and $U(H) = Z_a \otimes H + X_a \otimes \sqrt{1-H^2}$.

qubit (ancilla qubit)

$$U(H) = \begin{pmatrix} H & \cdot \\ \cdot & \cdot \end{pmatrix} \quad (\|H\| \leq 1)$$

$$U(H)^2 = I_a \otimes H^2 + I_a \otimes (I - H^2) + \{Z_a \otimes H, X_a \otimes \sqrt{I-H^2}\}$$

$$= I_a \otimes I + \underbrace{\{Z_a, X_a\}}_{=0} H\sqrt{I-H^2}$$

$$= I_a \otimes I$$

$$H = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$$

$$\sqrt{I-H^2} = \sum_\lambda \sqrt{1-\lambda^2}\, |\lambda\rangle\langle\lambda|$$

$$U(H)\, |0\rangle_a |\psi\rangle = (Z_a \otimes H)|0\rangle_a |\psi\rangle + X_a \otimes \sqrt{I-H^2}\, |\psi\rangle$$

$$= |0\rangle_a \otimes H|\psi\rangle + |1\rangle_a \otimes \sqrt{I-H^2}\, |\psi\rangle$$

$$U(H)\, |1\rangle_a |\psi\rangle = -|1\rangle_a \otimes H|\psi\rangle + |0\rangle_a \otimes \sqrt{I-H^2}\, |\psi\rangle$$

$|0\rangle_a|\psi\rangle, \quad |1\rangle_a|\psi\rangle$

$$
U(H) = \begin{array}{cc}
\begin{pmatrix} H & \sqrt{I-H^2} \\ \sqrt{I-H^2} & -H \end{pmatrix} & \begin{array}{l} |0\rangle|\psi\rangle \\ |1\rangle|\psi\rangle \end{array} \\
\begin{array}{cc} \underbrace{}_{|0\rangle|\psi\rangle} & \underbrace{}_{|1\rangle|\psi\rangle} \end{array} &
\end{array}
$$

Unitary Encoding

$U(H) = Z_a \otimes H + X_a \otimes \sqrt{1 - H^2}$ is called as a *unitary encoding* of H .

1. $U(H)$ is a unitary.
2. Alternatively, we can view it as a block-diagonal matrix:

$$U(H) = \begin{pmatrix} H & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

3. Of course, eventually we will need to discuss how to actually implement $U(H)$, given the Hamiltonian. (Hint: SELECT + PREPARE)

Qubitization: Energy Eigenstate

$$e^{i\phi_0 Z}e^{i\theta X}e^{i\phi_1 Z}\cdots e^{i\theta X}e^{i\phi_d Z} \to e^{i\phi_0'\tilde{Z}}U(H)e^{i\phi_1'\tilde{Z}}\cdots U(H)e^{i\phi_d'\tilde{Z}},$$

where $\tilde{Z} = Z_a \otimes I_s$ and $U(H) = Z_a \otimes H + X_a \otimes \sqrt{1 - H^2}$.

Let's first study the action of this operator on $|\lambda\rangle_s$, where $H|\lambda\rangle_s = \lambda|\lambda\rangle_s$.

$$e^{i\phi_0'\tilde{Z}}U(H)e^{i\phi_1'\tilde{Z}}|\lambda\rangle_s = e^{i\phi_0'\tilde{Z}}U(H)|\lambda\rangle_s e^{i\phi_1'\tilde{Z}}$$

$$= e^{i\phi_0'\tilde{Z}}(Z_a \otimes H + X_a \otimes \sqrt{I - H^2})|\lambda\rangle_s e^{i\phi_1'\tilde{Z}}$$

$$= e^{i\phi_0'\tilde{Z}}(Z_a\lambda + X_a\sqrt{1-\lambda^2})|\lambda\rangle_s e^{i\phi_1'\tilde{Z}}$$

$$= |\lambda\rangle_s e^{i\phi_0'\tilde{Z}}(Z_a\lambda + X_a\sqrt{1-\lambda^2})e^{i\phi_1'\tilde{Z}}$$

$$e^{i\phi_0'\tilde{Z}}U(H)e^{i\phi_1'\tilde{Z}}U(H)e^{i\phi_2'\tilde{Z}}|\lambda\rangle_s = e^{i\phi_0'\tilde{Z}}U(H)|\lambda\rangle_s e^{i\phi_1'\tilde{Z}}(Z_a\lambda + X_a\sqrt{1-\lambda^2})e^{i\phi_2'\tilde{Z}}$$

$$= e^{i\phi_0'\tilde{Z}}(Z_a\lambda + X_a\sqrt{1-\lambda^2})|\lambda\rangle_s e^{i\phi_1'\tilde{Z}}(Z_a\lambda + X_a\sqrt{1-\lambda^2})e^{i\phi_2'\tilde{Z}}$$

$$= e \quad (Z_a\lambda + X_a\sqrt{1-\lambda^2})\,|\lambda\rangle_S \quad (Z_a\lambda + X_a\sqrt{1-\lambda^2})$$

$$= |\lambda\rangle_S \, e^{i\phi_0'\tilde{Z}} (Z_a\lambda + X_a\sqrt{1-\lambda^2})\, e^{i\phi_1'\tilde{Z}} (Z_a\lambda + X_a\sqrt{1-\lambda^2})\, e^{i\phi_2'\tilde{Z}}$$

$$\sum_\lambda \alpha_\lambda |\lambda\rangle_S \rightarrow \sum_\lambda \alpha_\lambda |\lambda\rangle_S \, e^{i\phi_0'\tilde{Z}} R(\lambda)\, e^{i\phi_1'\tilde{Z}} \cdots R(\lambda)\, e^{i\phi_d'\tilde{Z}}$$

$$R(\lambda) = Z_a\lambda + X_a\sqrt{1-\lambda^2}$$

Qubitization: Relating to QSP

Qubitization: $e^{i\phi_0 Z} R(\lambda) e^{i\phi_1 Z} \cdots R(\lambda) e^{i\phi_d Z}$

Quantum Signal Processing: $e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z}$

$R(\lambda) = \lambda Z + \sqrt{1-\lambda^2} X$.

$e^{i\theta X} = I\cos(\theta) + i\sin(\theta) X$

How to relate the two?

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Clifford: $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ $\quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

phase gate $\quad$ Hadamard gate

$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$ $\qquad S^4 = I$

Clifford: U s.t. $UPU^\dagger =$ (possibly another) Pauli (up to a phase)

Pauli

$$SXS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

$$SYS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = -X$$

$$SZS^\dagger = Z$$

$$\begin{aligned} SR(\lambda)S &= S(\lambda Z + \sqrt{1-\lambda^2}\,X)\,S \\ &= \lambda(SZS) + \sqrt{1-\lambda^2}\,SXS \\ &= \lambda I + \sqrt{1-\lambda^2}\,SX\underbrace{S^\dagger S}_{I} S \\ &= \lambda I + \sqrt{1-\lambda^2}\,Y S^2 = \lambda I + \sqrt{1-\lambda^2}\,YZ \\ &= \lambda I + i\sqrt{1-\lambda^2}\,X \end{aligned}$$

$$\tilde{S} R(\lambda) \tilde{S} = \lambda I + i\sqrt{1-\lambda^2}\,X = e^{\frac{\pi}{4}iZ} R(\lambda)\, e^{\frac{\pi}{4}iZ}$$

$$e^{i\theta X} = I\cos\theta + i\sin\theta\, X \qquad \cos\theta = \lambda$$

$$S = e^{\frac{\pi}{4}iZ} \qquad \text{(up to a global phase)}$$

$$\boxed{R(\lambda) = e^{-\frac{\pi}{4}iZ}\, e^{iX\cos^{-1}\lambda}\, e^{-\frac{\pi}{4}iZ}}$$

Aborbing the phases

Qubitization: $e^{i\phi_0 Z} R(\lambda) e^{i\phi_1 Z} \cdots R(\lambda) e^{i\phi_d Z}$

Quantum Signal Processing: $e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z}$

$$R(\lambda) = \lambda Z + \sqrt{1-\lambda^2} X \,.$$

$$e^{i\theta X} = I\cos(\theta) + i\sin(\theta) X$$

$$\boxed{R(\lambda) = e^{-\frac{\pi}{4} i Z} \, e^{i X \cos^{-1}\lambda} \, e^{-\frac{\pi}{4} i Z}}$$

How to relate the two?

$$e^{i(\phi_0 - \frac{\pi}{4})Z} \, e^{i X \cos^{-1}\lambda} \, e^{i(\phi_1 - \frac{\pi}{4})Z} \cdots e^{i X \cos^{-1}\lambda} \, e^{i(\phi_d - \frac{\pi}{4})}$$

Qubitization vs. Quantum Signal Processing

$$e^{i\phi_0 Z}e^{i\theta X}e^{i\phi_1 Z}\cdots e^{i\theta X}e^{i\phi_d Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = \cos^{-1}(a)$. Thus, using qubitization, we can implement (upon measuring $|0\rangle$)

$$|\psi\rangle_s \rightarrow P(H)|\psi\rangle,$$

(for a polynomial that satisfies the conditions in QSP.)

$$e^{i\phi_0 \tilde{Z}} U(H) e^{i\phi_1 \tilde{Z}} \cdots U(H) e^{i\phi_d \tilde{Z}} |\lambda\rangle_s = e^{i\phi_0 \tilde{Z}} e^{i\theta \tilde{X}} e^{i\phi_1 \tilde{Z}} \cdots e^{i\theta \tilde{X}} e^{i\phi_d \tilde{Z}} |\lambda\rangle_s$$

$(\cos\theta = \lambda)$

$$e^{i\phi_0 \tilde{Z}} U(H) e^{i\phi_1 \tilde{Z}} \cdots U(H) e^{i\phi_d \tilde{Z}} |\lambda\rangle_s |0\rangle_a = P(\lambda)|\lambda\rangle_s|0\rangle_a + \cdots |\psi\rangle_s |1\rangle_a$$

$\downarrow$ measure a

If we measure $0 \rightarrow P(\lambda)|\lambda\rangle_s$

$$e^{i\phi_0 \tilde{Z}} U(H) e^{i\phi_1 \tilde{Z}} \cdots U(H) e^{i\phi_d \tilde{Z}} \sum_\lambda \alpha_\lambda |\lambda\rangle_s |0\rangle_a$$

$\downarrow$ measure a

$$0 \rightarrow \sum_\lambda P(\lambda) \alpha_\lambda |\lambda\rangle_s = P(H)|\psi\rangle$$

$$\left(|\psi\rangle = \sum_\lambda \alpha_\lambda |\lambda\rangle\right)$$

$$P(H) = \sum_\lambda P(\lambda) |\lambda\rangle\langle\lambda|$$

$U(H) \longrightarrow P(H)$ (probabilistically)

Qubitization vs. Quantum Signal Processing

Key point: Given any polynomial that satisfies the conditions in QSP, we can implement

$$|\psi\rangle_s \to \tilde{P}(H)|\psi\rangle_s.$$

Before the measurement: $P(H)|\psi\rangle_s|0\rangle_a + |\cdots\rangle$

$\Pr[\text{success}] = \langle\psi|_s P(H)^\dagger P(H)|\psi\rangle_s$

Qubitization & Quantum Signal Processing

$$e^{i\phi_0 Z}e^{i\theta X}e^{i\phi_1 Z}\cdots e^{i\theta X}e^{i\phi_d Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = \cos^{-1}(a)$.

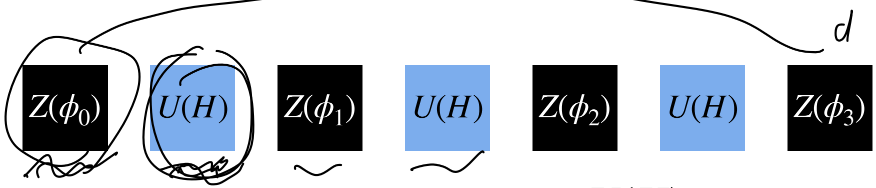
1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1$.

$P(H) \approx e^{-iHT}$

A more global picture

$$e^{i\phi_0'\tilde{Z}}U(H)e^{i\phi_1'\tilde{Z}}\cdots U(H)e^{i\phi_d'\tilde{Z}},$$

where $\tilde{Z} = Z_a \otimes I_s$ and $U(H) = Z_a \otimes H + X_a \otimes \sqrt{1 - H^2}$.



Cost = Cost of $U(H)$ $\times$ Degree of the polynomial

Hamiltonian Simulation

Key question: How to find a polynomial approximation of
$e^{-iHt} = \cos(Ht) - i\sin(Ht)$?

Fortunately, there is already a wealth of literature on this matter.

Chebyshev Polynomial (first kind)

Definition: $T_n(x)$ such that $T_n(\cos\theta) = \cos(n\theta)$.

This is frequently used in approximating arbitrary functions.

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin\alpha\sin(\beta)$

Jacobi-Anger expansion

Taylor expansion
$O(|t| \times \text{poly}(\log \frac{1}{\varepsilon}))$

$$\cos(xt) = J_0(t) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(t) T_{2k}(x)$$

$$\sin(xt) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(t) T_{2k+1}(x)$$

To achieve ϵ error, it suffices to choose to truncate the polynomial at $O\left(|t| + \frac{\log(1/\epsilon)}{\log\left(e + \frac{\log(1/\epsilon)}{|t|}\right)}\right)$'th order. [Low and Chuang (2016), Gilyen, Su, Low, and Wiebe (2019)]

Success Probability

Success probability = norm of $P(H)|\psi\rangle$. Thus, if $P(H)$ is a unitary, the success probability is 1.

Even for Hamiltonian simulation, because we truncate the polynomial to some finite degree, $P(H)$ will not be exactly unitary.

But that's okay, because (i) we can make $P(H)$ "as unitary as we want" by making the degree larger.

P, Q

$$\underline{|P|^2 + (1-x^2)|Q|^2 = 1}$$

Remaining questions

Unitary encoding of H

1. How do we implement the unitary oracle?
2. Examples?
3. Next time!

$H = a^\dagger a$ $\quad$ $\mathrm{Spec}(H) = \{0, 1, 2, \ldots \infty\}$